PVDIS Mass Limit Contours

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We describe how to derive the mass limit contour, provided we have PVDIS and elastic PVES data (such as those from SoLID deuteron and P2) that constrain the couplings g^{eq}_{VA} and g^{eq}_{AV}.

FORMALISM

For convenience, we show the low-energy effective Lagrangian of the weak neutral-current interaction relevant to electron deep inelastic scattering (DIS) off nuclear targets below:

$$L^{eq}_{NC} = \frac{G_F}{\sqrt{2}}\sum_q \left[g^{eq}_{VV}\bar{e}\gamma^\mu e\bar{q}\gamma_\mu q + g^{eq}_{AV}\bar{e}\gamma^\mu\gamma_5 e\bar{q}\gamma_\mu q + g^{eq}_{VA}\bar{e}\gamma^\mu e\bar{q}\gamma_\mu\gamma_5 q + g^{eq}_{AA}\bar{e}\gamma^\mu\gamma_5 e\bar{q}\gamma_\mu\gamma_5 q\right], \tag{1}$$

where G_F is the Fermi constant. At lowest order (tree level) in the Standard Model (SM), the four-fermion couplings are products of the lepton and quark couplings $g^f_{V,A}$ to the Z boson,

$$g^{eu}_{AV} = 2g^e_A g^u_V = -\frac{1}{2} + \frac{4}{3}\sin^2\theta_W\ , \tag{2}$$

$$g^{ed}_{AV} = 2g^e_A g^d_V = +\frac{1}{2} - \frac{2}{3}\sin^2\theta_W\ , \tag{3}$$

$$g^{eu}_{VA} = 2g^e_V g^u_A = -\frac{1}{2} + 2\sin^2\theta_W = -g^{ed}_{VA} = -2g^e_V g^d_A \tag{4}$$

where θ_W is the weak mixing angle.

PVDIS DATA CONSTRAINTS

Data on the PVDIS asymmetry on the deuteron will provide constraints in the specific form of

$$A^{\rm DIS}_{RL,d} \approx \frac{3}{20\pi\alpha}\frac{Q^2}{v^2}\left[(2g^{eu}_{AV} - g^{ed}_{AV}) + (2g^{eu}_{VA} - g^{ed}_{VA})R_V\frac{1-(1-y)^2}{1+(1-y)^2}\right], \tag{5}$$

where α is the fine structure constant and $v = (\sqrt{2}G_F)^{-1/2} = 246.22$ GeV is the Higgs vacuum expectation value setting the electroweak scale. We simplify this to

$$A_{RL,d} = a\left[C_1 + YC_2\right] \tag{6}$$

where Q^2 is absorbed into a and R_V is absorbed into Y. The couplings are abbreviated to $C_1 = 2g^{eu}_{AV} - g^{ed}_{AV}$ and $C_2 = 2g^{eu}_{VA} - g^{ed}_{VA}$.

Once some data are collected, ideally more than one point, a χ^2 can be defined as

$$\chi^2 = \sum_k\left[\frac{a(k)C_1 + a(k)Y(k)C_2 - A^{\rm data}_{PV}(k)}{\delta A^{\rm data}_{PV}(k)}\right]^2 \tag{7}$$

where the summation k is over all available data points and $C_{1,2}$ are SM couplings (with the appropriate isospin combination) to be fitted. The best fit values $\bar{C}_1$ and $\bar{C}_2$ are determined by

$$\left.\frac{\partial\chi^2}{\partial C_i}\right|_{C_i=\bar{C}_i} = 0\ . \tag{8}$$

Taking the derivative, we have:

$$2\sum_k\frac{\left(a(i)\bar{C}_1 + a(k)Y(k)\bar{C}_2 - A^{\rm data}_{PV}(k)\right)a(k)}{\left(\delta A^{\rm data}_{PV}(k)\right)^2} = 0 \tag{9}$$

$$2\sum_k\frac{\left(a(i)\bar{C}_1 + a(k)Y(k)\bar{C}_2 - A^{\rm data}_{PV}(k)\right)a(k)Y(k)}{\left(\delta A^{\rm data}_{PV}(k)\right)^2} = 0\ . \tag{10}$$

In matrix form, the above two equations become:

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \bar{C}_1 \\ \bar{C}_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} , \tag{11}$$

from which we can obtain the best fit values $\bar{C}_1, \bar{C}_2$. Note that elastic PVES experiments provide data on C_1 alone. We can add this to the χ^2 definition as an additional term, but the matrix formate of Eq. (11) would remain the same.

The uncertainty of the fitted values $\bar{C}_1, \bar{C}_2$ are obtained by expanding χ^2 around the best-fit values and require that

$$\chi^2(C_1, C_2) - \chi^2(\bar{C}_1, \bar{C}_2) \leqslant K \tag{12}$$

for some constant K (that corresponds to the desired C.L. value). The commonly displayed ellipse constraint is given by

$$S_{11}(C_1 - \bar{C}_1)^2 + 2S_{12}(C_1 - \bar{C}_1)(C_2 - \bar{C}_2) + S_{22}(C_2 - \bar{C}_2)^2 \leqslant K \ . \tag{13}$$

Alternatively, the ellipse is given by the uncertainty in each of C_1, C_2, and a correlation factor. For example, PDG has for the current world data:

$$(\Delta C_1)_{\text{world}} = 0.0068,\ (\Delta C2)_{\text{world}} = 0.06, \text{ with correlation } (corr_{12})_{\text{world}} = -0.22.$$

The PDG central values of $C_{1,2}$ have slight offsets from SM values, which we omit here. Similarly, the projected uncertainties using SoLID deuteron measurement (full beam time, full data coverage):

$$(\Delta C_1)_{\text{solid}} = 0.03353,\ (\Delta C_2)_{\text{solid}} = 0.04046, \text{ with correlation } (corr_{12})_{\text{solid}} = -0.986.$$

And if also including P2 projected results:

$$(\Delta C_1)_{\text{solid+P2}} = 0.00235,\ (\Delta C_2)_{\text{solid+P2}} = 0.00729, \text{ with correlation } (corr_{12})_{\text{solid+P2}} = -0.38.$$

All SoLID projected uncertainties above (with or without P2) are using the full statistical, experimental systematic, CJ15 PDF (which provides the smallest PDF uncertainty), and with the hadronic physics terms (β_{HT} and β_{CSV}) fitted simultaneously with the $C_{1,2}$. Of course, for a 4-parameter fit the error matrix has dimension 4×4. The correlation value is from the 2×2 submatrix that corresponds to $C_{1,2}$.

NEW PHYSICS MASS LIMIT

The energy scale, Λ, up to which new physics Beyond the SM (BSM) is testable, can be quantified in terms of perturbations of the SM Lagrangian (1), that is by replacements of the form,

$$\frac{G_F}{\sqrt{2}} g_{ij} \to \frac{G_F}{\sqrt{2}} g_{ij} + \eta^q_{ij} \frac{4\pi}{(\Lambda^q_{ij})^2} , \tag{14}$$

where $ij = AV, VA, AA$ and we assumed that the new physics is strongly coupled with a coupling g given by $g^2 = 4\pi$.

Now we want to plot $\Lambda_{AV,VA}$ on a 2D plane, with the understanding that Λ_{AV} comes from $2g^{eu}_{AV} - 2g^{ed}_{AV}$ and Λ_{VA} from $2g^{eu}_{VA} - 2g^{ed}_{VA}$. However, the two sets of couplings are not independent quantities when constrained using PVES data, and thus we define a radial plot where the radius Λ represents the BSM mass scale regardless of the nature of such BSM model (i.e. which coupling it affects), and a phase angle α that points towards which direction (AV vs. VA) the BSM physics alters. To do so, we start from the general description of an ellipse, illustrated in Fig. 1). All points along the contour of the ellipse can be described using the rotated frame as:

$$x' = a \cos t, \tag{15}$$

$$y' = b \sin t \tag{16}$$

where $t \in (0, 2\pi)$ and furthermore to (x, y):

$$x = x' \cos\theta_0 - y' \sin\theta_0 = a \cos t \cos\theta_0 - b \sin t \sin\theta_0, \tag{17}$$

$$y = x' \sin\theta_0 + y' \cos\theta_0 = a \cos t \sin\theta_0 + b \sin t \cos\theta_0 \ . \tag{18}$$

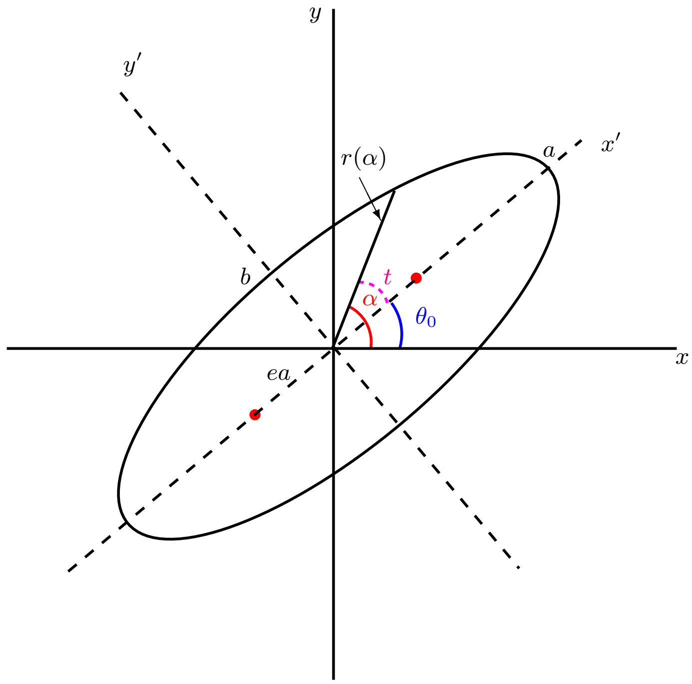

FIG. 1. General description of an ellipse. The ellipse is centered at $(0, 0)$, with half length of semi-major axis a, half length of semi-minor axis b, ellipticity $e = \sqrt{1 - b^2/a^2}$, and tilt angle θ_0 w.r.t. the horizontal axis. The red dots show the location of foci points, which are at distance ea from the ellipse center.

The uncertainty limit on the $C_{1,2}$ in the BSM physics direction $\alpha \equiv t + \theta_0$ is

$$r(\alpha) = \sqrt{x^2 + y^2} \ , \tag{19}$$

which corresponds to BSM mass limit:

$$\Lambda(\alpha) = \frac{0.24622\sqrt{8\pi}}{\sqrt{r(\alpha)}} \text{ in TeV} \ . \tag{20}$$

The mass limits for the C_1 and C_2 directions are

$$\Lambda_1 = \Lambda \cos\alpha \tag{21}$$

$$\Lambda_2 = \Lambda \sin\alpha \ , \tag{22}$$

which can be visualized (plotted) as a contour.

The only remaining technical detail is to relate the fitted uncertainty matrix to the semi-major and semi-minor axis lengths and tilt angle. For this we follow https://cookierobotics.com/007/ but re-express them in our notation as:

$$\sigma_{11} = (\Delta C_1)^2, \quad \sigma_{12} = (corr_{12})(\Delta C_1)(\Delta C_2), \quad \sigma_{22} = (\Delta C_2)^2 \ , \tag{23}$$

which corresponds to axis half-lengths:

$$a = \sqrt{\frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2}{4} + \sigma_{12}^2}}; \tag{24}$$

$$b = \sqrt{\frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2}{4} + \sigma_{12}^2}}; \tag{25}$$

and tilt angle:

$$\theta_0 = atan2(a^2 - \sigma_{11}, \sigma_{12}) \ . \tag{26}$$

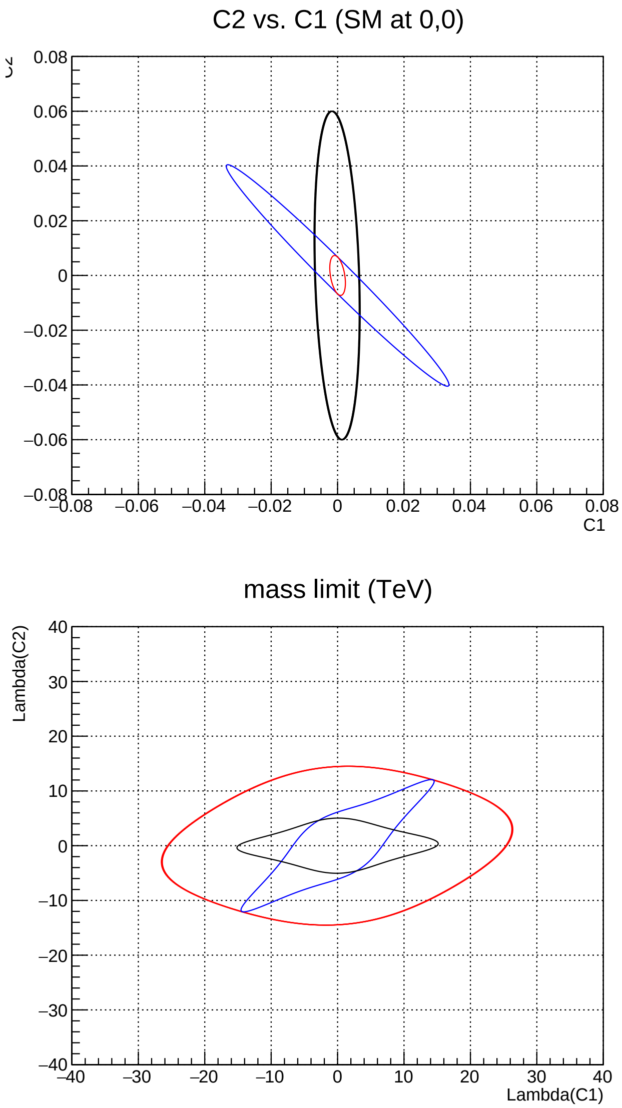

FIG. 2. Top: Uncertainty ellipse on $C_{1,2}$ from PDG world data (black), projected SoLID deuteron measurement (blue), and projected SoLID deuteron and P2 measurements combined (red). Bottom: Resulting mass limit contour with the same color coding. Note that we have neglected to plot the SM values as well as any deviation of world data from the SM value.

Scanning $t \in (0, 2\pi)$ yields the mass limit contour in Fig. 2, which looks rather similar to the plots that Jens has always been making.

Discussions:

One may wonder why the mass limit contour appears to be a "pinched" ellipse. In fact, one can prove that if the $C_{1,2}$ constraint were a circle, the resulting mass limit would also be a circle. However, if the $C_{1,2}$ constraint were an ellipse, then the mass limit in any direction not along the semi-major or semi-minor axis of the ellipse would be smaller than when an ellipse is drawn directly on the mass limit plane using the given axis lengths. Intuitively, one could say that this is a direct outcome from the fact that the mass limit is inverse of $\sqrt{\Delta C}$, but a strict proof would require some math knowledge that I do not have. Similarly, I had naively expected that one could plot the contour analytically, that is, directly from the χ^2 definition. This method is detailed in the appendix, which turned out to not work. Other methods were also tried, such as forming the uncertainty of the $C_{1,2}$ combination in the BSM direction α using error propagation with $C_{1,2}$ error matrix as input, and then take the inverse, but it also does not provide the same contour.

Expected Contour

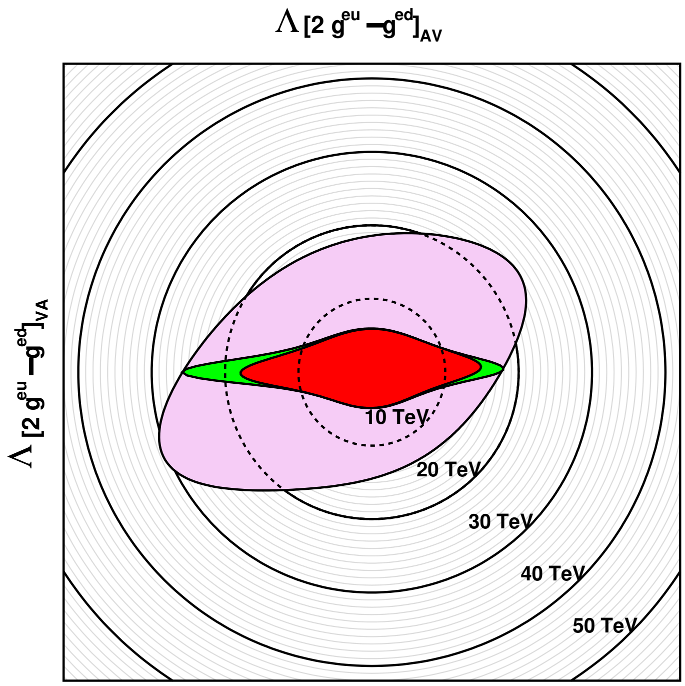

FIG. 3. Expected mass limit contour (red), from Jens.

METHODS THAT DID NOT WORK

We have tried several alternate ways, attempting to produce the mass limit contour analytically. We write

$$C_1 \rightarrow C_1^{SM} + \frac{4\pi}{\Lambda^2}\cos\alpha \ , \tag{27}$$

$$C_2 \rightarrow C_2^{SM} + \frac{4\pi}{\Lambda^2}\sin\alpha \ . \tag{28}$$

In principle, we should be able to transform the χ^2 equation Eq.(12) from expressions of $C_{1,2}$ to expressions of Λ, α, which should help us visualize the mass limit contour. As a starting point, we can assume the minimal χ^2 value $\chi^2(\bar{C}_1, \bar{C}_2)$ to be a constant fixed by the data set and absorb it into K:

$$\chi^2 \leqslant K' \tag{29}$$

and we rewrite the original χ^2 definition, Eq. (7), in terms of Λ and α. Again as a starting point, we assume $\bar{C}_{1,2}$ coincide with the SM values:

$$\begin{aligned}\chi^2 &= \sum_k \left[\frac{a(k)C_1 + a(k)Y(k)C_2 - A_{PV}^{\rm data}(k)}{\delta A_{PV}^{\rm data}(k)}\right]^2 \\ &= \sum_k \left[\frac{a(k)\left(C_1^{SM} + \frac{4\pi\cos\alpha}{\Lambda^2}\right) + a(k)Y(k)\left(C_2^{SM} + \frac{4\pi\sin\alpha}{\Lambda^2}\right) - A_{PV}^{\rm data}(k)}{\delta A_{PV}^{\rm data}(k)}\right]^2 \qquad (30) \\ &\leqslant K' \ . \qquad (31)\end{aligned}$$

We can define how the data deviate from the SM value as $e_k = A_{PV}^{\rm data}(k) - a(k)\left(C_1^{SM} + Y(k)C_2^{SM}\right)$, the relative precision of the measurement $\delta A_{PV}^{\rm data}(k) = d(k)A_{PV}^{\rm data}$ and write

$$\chi^2 = \sum_k \left[\frac{a(k)\frac{4\pi\cos\alpha}{\Lambda^2} + a(k)Y(k)\frac{4\pi\sin\alpha}{\Lambda^2} - e(k)}{d(k)A_{PV}^{\rm data}(k)}\right]^2 \leqslant K' \ . \tag{32}$$

And then we write $\Lambda_x = \Lambda\cos\alpha$ and $\Lambda_y = \Lambda\sin\alpha$ and plot these as (x, y). To write all these in one analytic equation, we replace the Λ by $\sqrt{x^2+y^2}$, $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$, resulting in one final equation:

$$\chi^2 = \sum_k \left[\frac{a(k)\frac{4\pi x}{(x^2+y^2)^{3/2}} + a(k)Y(k)\frac{4\pi y}{(x^2+y^2)^{3/2}} - e(k)}{d(k)A_{PV}^{\rm data}(k)}\right]^2 \leqslant K' \ . \tag{33}$$

However, coding this in Desmos (see Fig. 4) did not give us the desired shape. Instead, it has a "peanut" shape with the direction with the least constraint in $C_{1,2}$ the pinched side of the peanut, see Fig. 5. I also experimented with modifying Eq. (33) to:

$$\chi^2 = \sum_k \left[\frac{a(k)\frac{4\pi|x|}{(x^2+y^2)^{3/2}} + a(k)Y(k)\frac{4\pi|y|}{(x^2+y^2)^{3/2}} - e(k)}{d(k)A_{PV}^{\rm data}(k)}\right]^2 \leqslant K' \ . \tag{34}$$

even though such modification is not justified. The output is shown in orange color in Fig. 5, which still does not present similarity to the PVDIS mass limit contour.

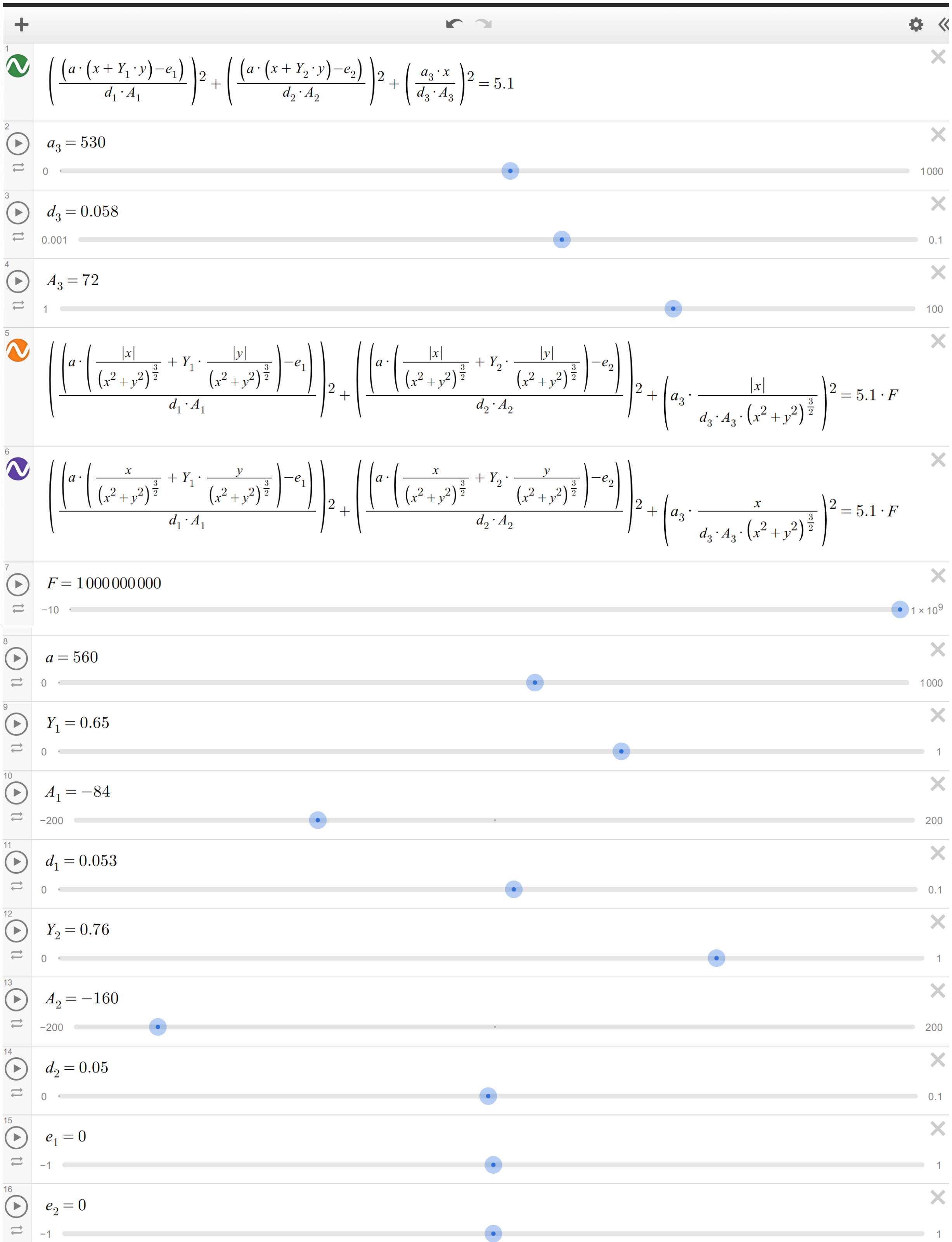

FIG. 4. Toy model parameters. The three cells 1, 6 and 5 are for Eqs. (12), (33) and (34), and are shown in green, purple, and orange, respectively. The parameter F is for scaling such that all three can be viewed on the same graph. The green contour is supposed to mimic the $C_{1,2}$ ellipse and the purple and orange contours are the resulting mass limits.

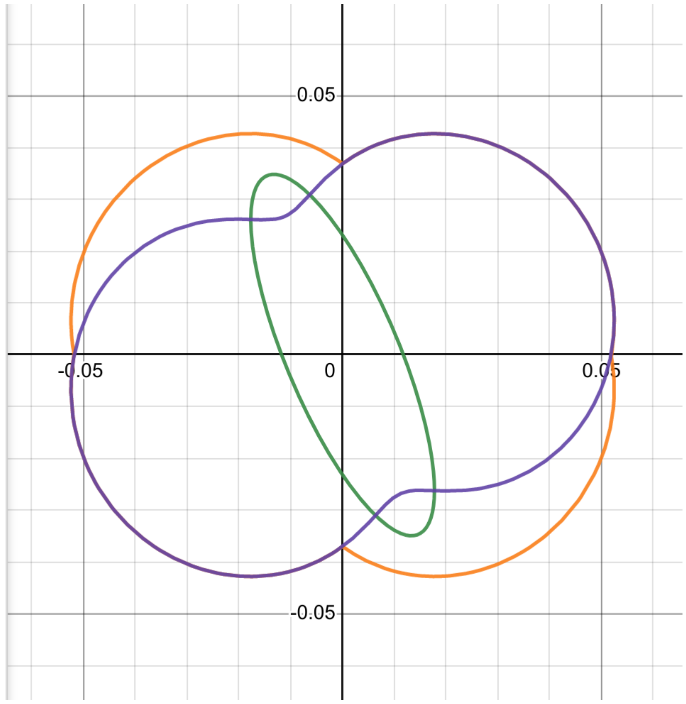

FIG. 5. Toy model output. The green contour represents the uncertainty bound for $C_{1,2}$ from PVDIS-type and P2-type data combined, while purple, and orange represent mass limit contours as described by Eqs. (33) and (34), respectively.